# Analytical Model for the Calculation of Long Term Crack Width

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*Abstract*—**Cracking is one of the important limit states to be considered in the design of concrete structures. When a flexural member is subjected to a load greater than or equal to the cracking load, the width of cracks increases with time. For controlling the width and spacing of cracks to an acceptable limit the designer should be able to predict the maximum crack width under sustained loading. This paper provides a relatively simple and accurate analytical model for the prediction of maximum width of cracks of reinforced cement concrete flexural members subjected to long-term loading. The proposed model is demonstrated through a numerical example. The model is compared with the test results available in the literature.**

***Keywords—bond stress; cracking moment; flexural crack; tension stiffening.***



## I. INTRODUCTION

Cracks can be load dependent or independent. Load dependent cracks are due to flexure, shear, tension, torsion and combined effects of any of these. Cracks due to creep, shrinkage, corrosion, weathering and chemical actions can be considered as time dependent or load independent cracks. When a reinforced concrete beam sustains a load equal to or greater than the cracking load, the width of cracks increases with time. The properties of concrete like creep, shrinkage and tension stiffening influence the increase in crack width. Many codes of practice specify maximum steel stress increments after cracking and maximum spacing requirements for the reinforcement. However, few existing code procedures account adequately for the gradual increase in existing crack widths with time due to creep, shrinkage, sustained loading and tension stiffening. The determination of the time varying strains, stresses and curvatures at critical sections of reinforced concrete members is an important requisite for their serviceability analysis and design. Creep under sustained loads and shrinkage result in redistribution of strains and stresses between steel and adjoining concrete. Consequently, a continuous change of the depth of neutral axis occur leading to a time varying effective compression area. This character is the reason for the complex nature of analysis of reinforced concrete sections under sustained loads.

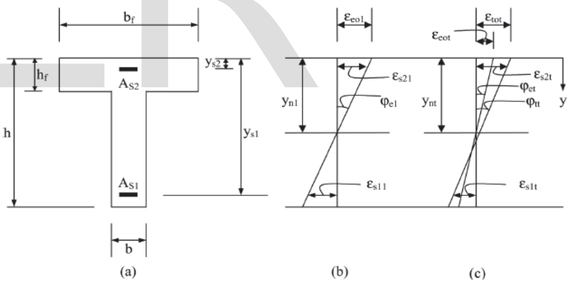
This paper presents a simple analytical model for prediction of time varying crack width of reinforced concrete flexural members under sustained loading. The model is derived based on equilibrium and compatibility conditions.

## II. ANALYTICAL MODEL FOR CRACKWIDTH

Throughout the formulation of the proposed model, compressive forces, stresses and the corresponding deformations are assumed to be positive. Positive bending moments and curvatures are those producing tension in the bottom fibres of the cross section.

### *A. Cracking Moment*

A typical R. C. flanged section considered for the model formulation is given in Fig. 1(a).



(a) R.C. section (b) Strain at time $t_1$ (c) Strains at time t

Fig. 1 R.C.C section and schematical illustration for strain distribution

| | |
|---|---|
| $A_s$ | Area of steel |
| $h$ | overall depth of section |
| $y_n$ | depth of neutral axis from the farthest compression fibre |
| $y_{si}$ | depth of reinforcement from the farthest compression fibre |
| $\Delta \varepsilon_s$ | reduction in steel strain due to tension stiffening effect |
| $\varepsilon_{eo1}$ | elastic strain in the farthest compressed fibre at time $t_1$ |
| $\varepsilon_{eot}$ | elastic strain in the farthest compressed fibre at time t |
| $\varepsilon_{ey1}$ | elastic strain in concrete at a distance y from the farthest compressed fibre at time $t_1$ |
| $\varepsilon_{eyt}$ | elastic strain in concrete at a distance y from the farthest compressed fibre at time t |
| $\varepsilon_s$ | strain in steel |
| $\varepsilon_{s1cr}$ | strain in steel at uncracked section under cracking moment |
| $\varepsilon_{s2}$ | strain in steel at cracked section under design moment |
| $\varepsilon_{s2cr}$ | strain in steel at cracked section under cracking moment |
| $\varepsilon_{tot}$ | total strain in the farthest compressed fibre at time t |

$\varepsilon_{tyt}$ total strain in concrete at a distance y from the farthest compressed fibre at time t
$\psi_{e1}$ elastic curvature at time $t_1$
$\psi_{et}$ elastic curvature at time t
$\psi_{tt}$ total curvature at time t

Strain distributions at time $t_1$, when the loads are initially applied is shown in Fig 1 (b) and strain at a later time t is shown in Fig. 1 (c). Al-Zaid [1] developed an analytical model for the calculation of long term cracking moment. The strains at time 't' are calculated considering creep, aging coefficient and shrinkage strain as reported in the paper by Al Zaid[1].

The equilibrium equations at any instant *t* can be written in the general matrix form as given in "(1)"

$$\begin{Bmatrix} \overline{P} \\ \overline{M} \end{Bmatrix} = E_c \begin{bmatrix} A_i & -S_i \\ -S_i & I_i \end{bmatrix} \begin{Bmatrix} \varepsilon_{e0t} \\ \psi_{et} \end{Bmatrix} \tag{1}$$

Where $A_i$ = Area of concrete under compression, $E_c$ = modulus of elasticity of concrete, $\overline{M}$ = Resultant moment, $\overline{P}$ = Net axial force, $S_i$ = First moment of $A_i$ about the farthest compressed fibre. $I_i$ = Second moment of $A_i$ about the farthest compressed fibre.

When the stress in concrete at extreme tension fibre (bottom most fibre) is equal to the modulus of rupture, strain at extreme compression fibre (top most fibre) is given by "(2)"

$$\varepsilon_{e01} = \frac{f_r}{E_c} + \psi_{e1}.h \tag{2}$$

Where, *h* is the height of the section and *fr is* the modulus of rupture of concrete. When the stress in concrete at extreme tension fibre (bottom most fibre) is equal to the modulus of rupture at time $t_1$, moment will become the cracking moment.

Combining "(1)" & "(2)"

$$\begin{Bmatrix} \overline{P} \\ M_{cr} \end{Bmatrix} = E_c \begin{bmatrix} A_1 & -S_1 \\ -S_1 & I_1 \end{bmatrix} \begin{Bmatrix} \frac{f_r}{E_c} + \psi_{e1}.h \\ \psi_{e1} \end{Bmatrix} \tag{3}$$

Solving "(3)", the expression for cracking moment is obtained as

$$M_{cr1} = -f_r S_1 + \left(\overline{P} - f_r A_1\right)\left(\frac{I_1 - S_1.h}{A_1.h - S_1}\right) \tag{4}$$

*B . Bond Stress*

The bond force acting at the interface of reinforcement bar and the surrounding concrete has great influence on the initiation and propagation of flexural cracks (4). The action of the bond stress is the main factor contributing to the tension stiffening effect in concrete structures. Peak bond stress occurs at the mid section between the two zero points, with a parabolic variation as shown in Fig. (2).

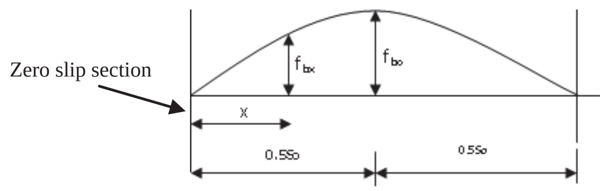



Fig.2. Variation of bond stress between zero-slip section and cracked section

The bond stress at any point *x* away from the zero slip section is given in "(5)"

$$f_{bx} = 4 f_{bo} \frac{x}{S_o}\left(1 - \frac{x}{S_o}\right) \tag{5}$$

Where $f_{bo}$ = peak bond stress, $f_{bx}$ = bond stress at any distance x from zero slip point and $S_o$ = Minimum crack spacing.

*C. First Flexural Crack*

When the resulting moment is equal to the cracking moment of a beam first crack appears on a beam. When the first flexural crack is developed in a loaded beam, the steel stress at the cracked section increases as the tensile force carried by concrete is transferred to the steel bars. This steel stress increment is resisted by bond forces developed along a certain length of the steel bars on either side of the crack.

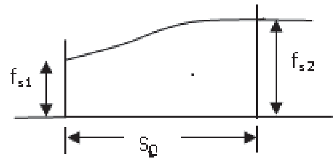



Fig.3 Variation of steel stress along the slip length

Variation of the steel stress along the slip length is shown in Fig. 3. The distance $S_o$ is herein referred to as the slip length. The steel stress increment due to the formation of the crack has resulted in a slip for a distance $S_o$ as shown in Fig. 3. Stresses and strain conditions at sections where slip has not taken place are assumed to be unchanged due to the formation of first crack. The steel stress $f_{s1}$ can be calculated using the procedure for an uncracked section. The steel stress $f_{s2}$ at the cracked section is evaluated using the procedure for a cracked section. The total change in force in steel is given by "(6)"

$$F_s = A_{st}\left(f_{s2} - f_{s1}\right) \tag{6}$$

The total bond force transferred is given by "(7)"

$$F_b = \int_0^{S_o} 4 f_{bo} \frac{x}{S_o}\left(1 - \frac{x}{S_o}\right) P_{st}\, dx \tag{7}$$

Taking equal diameter bars the expression for slip length will become as given in "(8)"

$$S_o = \frac{3\varphi}{8 f_{bo}}\left(f_{s2} - f_{s1}\right) \tag{8}$$

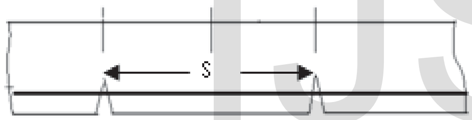
Since the crack has reduced the concrete surface stress to below the tensile strength of the concrete within ± $S_o$ of the crack, the next crack to form must form outside this region. The minimum distance between the cracks is thus $S_o$.

If cracks form at a lesser spacing than $2S_o$, the concrete stresses will be reduced over the whole length between the two cracks and another crack will not form. The maximum spacing will thus be $2S_o$. For the final stabilized crack pattern crack spacing $S$ is commonly been assumed to be 1.33 $S_o$ which is given in "(9)"

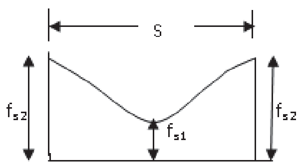
$$S = \frac{\varphi}{2 f_{bo}}\left(f_{s2cr} - f_{s1cr}\right) \tag{9}$$

### *D. Adjacent Flexural Crack*

Fig.4 (a) shows part of a beam with stabilised flexural crack pattern. The steel stress increment due to the formation of the crack has resulted in a slip for a distance $S$ as shown. Variation of the steel stress along the slip length is shown in Fig. 4(b).




(a) Adjacent flexural cracks in a beam




(b) Variation of steel stress between the two cracks

Fig. 4 Adjacent cracks in a beam

The condition of equilibrium of forces acting on part of the steel bar between the cracked section and the zero- slip point is given in "(10)"

$$A_{st}\left(f_{s2} - f_{s1}\right) = \int_0^{0.5S} 4 f_{bo} \frac{x}{0.5S}\left(1 - \frac{x}{0.5S}\right) P_{st}\, dx \tag{10}$$

$$f_{s2} - f_{s1} = \frac{4}{3\varphi} f_{bo} S \tag{11}$$

### *E. Tension Stiffening Effect*

For a reinforced concrete flexural member subjected to a bending moment $M > M_{cr}$ cracking occurs and the steel stress along the reinforcement varies from a maximum value at the crack location to a minimum value at the middle of the spacing between the cracks [(2)]. It is assumed that the concrete between the cracks has some effect on the mean strain in steel.

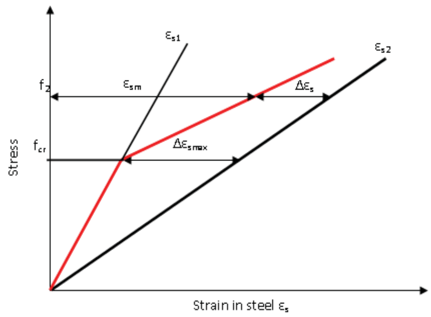



Fig.5. Tension stiffening effect

$\Delta\varepsilon_s$ is the reduction in steel strain caused by the participation of concrete in carrying the tensile stress between the cracks (tension stiffening effect). $\Delta\varepsilon_s$ will be maximum at the cracking moment. Based on experimental evidence it is assumed that $\Delta\varepsilon_s$ has hyperbolic variation. $\Delta\varepsilon_s$ at any point is given by " (12)"

$$\Delta\varepsilon_s = \Delta\varepsilon_{s\max} \frac{f_{sr}}{f_{s2}} \tag{12}$$

From the geometry of the figure Fig. 5 we get "(13)" and "(14)"

$$\Delta\varepsilon_{s\max} = \left(\varepsilon_{s2} - \varepsilon_{s1}\right)\frac{f_{sr}}{f_{s2}} \tag{13}$$

$$\Delta\varepsilon_{s\max} = \frac{\left(f_{s2} - f_{s1}\right)}{E_s}\frac{f_{sr}}{f_{s2}} \tag{14}$$

Substituting " (11)", "(14)", in "(12)" we get a new equation for $\Delta\varepsilon_s$ at any point as given in "(15)"

$$\Delta\varepsilon_s = \frac{4 f_{bo} S}{3\varphi E_s}\left(\frac{f_{sr}}{f_{s2}}\right)^2 \tag{15}$$

Strain in concrete at any distance x from the zero slip point is given by "(16)"

$$\Delta\varepsilon_{sx} = 4\Delta\varepsilon_s \frac{x}{S}\left(1 - \frac{x}{S}\right) \tag{16}$$

Substituting for $\Delta\varepsilon_s$ " (15)" in " (16)" strain in concrete at any distance x from the zero slip point is given by " (17)".

$$\Delta\varepsilon_{sx} = \frac{4 f_{bo} S}{3\varphi E_s}\left(\frac{f_{sr}}{f_{s2}}\right)^2 4\frac{x}{S}\left(1 - \frac{x}{S}\right) \tag{17}$$

Total extension in concrete, $e_c$ is calculated by integrating the strain function over the slip length and finally obtained the expression as given in " (18)".

$$e_c = \frac{4 f_{bo} S^2}{9 \varphi E_s} \left( \frac{M_{cr}}{M} \right)^2 \qquad (18)$$

*F. Crack Width*

Crack width at the level of reinforcement is determined as the relative difference in elastic extensions of steel and surrounding concrete. Both extensions are measured with respect to zero-slip point. The extension in steel is the product of spacing of crack and strain in steel. Crack width at the level of reinforcement is given by "(19)".

$$w_s = S\varepsilon_s - 2e_c \qquad (19)$$

Substituting for $S$, $\varepsilon_s$ & $e_c$ , Crack width at the level of reinforcement is given by " (20)".

$$w_s = \frac{\varphi E_s}{2 f_{bo}} \left( \varepsilon_{s2cr} - \varepsilon_{s1cr} \right) \left[ \varepsilon_{s2} - \frac{4}{9} \left( \varepsilon_{s2cr} - \varepsilon_{s1cr} \right) \left( \frac{M_{cr}}{M} \right)^2 \right] \qquad (20)$$

The maximum crack width at the tension face of the member is given in "(21)".

$$w_{\max} = \frac{h - y_n}{d - y_n} w_s \qquad (21)$$

Substituting "(20)" in "(21)", " (22)" is obtained.

$$w_{s\max} = \left( \frac{h - y_n}{d - y_n} \right) \times \frac{\varphi E_s}{2 f_{bo}} \left( \varepsilon_{s2cr} - \varepsilon_{s1cr} \right) \left[ \varepsilon_{s2} - \frac{4}{9} \left( \varepsilon_{s2cr} - \varepsilon_{s1cr} \right) \left( \frac{M_{cr}}{M} \right)^2 \right] \qquad (22)$$

## III. NUMERICAL EXAMPLE

The model proposed is applicable for both rectangular and flanged sections, with or without compression steel. The proposed model takes into account the change in effective area of concrete by allowing the movement of neutral axis, creep and shrinkage of concrete. The following numerical example will provide the demonstration of the model.

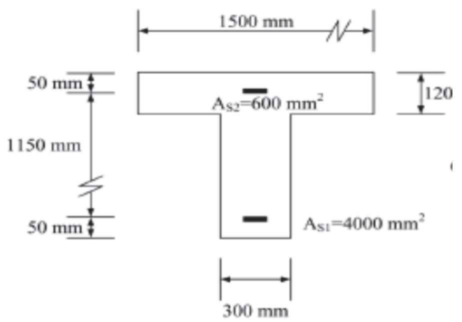



Fig. 6 Reinforced concrete section used for illustration

The T section as shown in Fig. 6 is subjected to an axial load $P$ = 200 kN at $e$ = 1000 mm from the farthest compressed fibre and a bending moment due to service sustained loads, $M_w$ = 1000 kNm. $Es$ = 200GPa, $\chi$ = 0.75, $M_{20}$ mix concrete and Fe 415 steel. Variation of crack width with time and corresponding percentage variation is given in Table 1. At the initial stage, the percentage of increase in width of crack is more. Width of crack increases gradually with time later at a slow rate. At t = 1year the percentage increase is 39.12 and at t = 2 years, it becomes 40.42.

TABLE 1.Variation of crack width with time

| **Time** | **crack width (mm)** | **percentage increase in crack width** |
|---|---|---|
| Initial | 0.0872 | |
| 1 year | 0.1213 | 39.119 |
| 2 year | 0.1224 | 40.424 |
| 3 year | 0.1228 | 40.792 |
| 4 year | 0.1229 | 40.948 |
| 5 year | 0.1230 | 41.028 |

## IV. COMPARISON WITH THE TEST RESULTS

The Fig.7 & 8 show the predicted variation in crack width with time and the experimental values of singly reinforced beam of Lutz[3]. From Fig. 7 & 8, it is clear that the proposed model gives good results at all stages. Both bottom crack width and side crack width are compared here. The average value of ratio of predicted width of crack to the experimental value is obtained as 0.963.

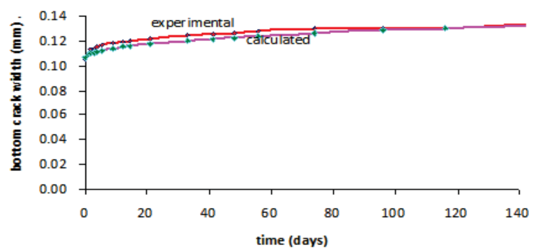



Fig. 7 Comparison of proposed model with test results of Lutz (Singly reinforced beam-Bottom crack width)

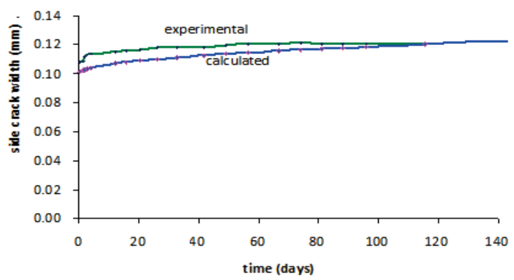



Fig. 8 Comparison of proposed model with test results of Lutz (Singly reinforced beam - Side crack width)

The Fig.9 &10 show the predicted variation in crack width with time and the experimental values of doubly reinforced beam of Lutz[3]. From Fig.9 & 10 it is clear that the proposed model gives comparable results at all stages. Both bottom crack width and side crack width are compared here. The average value of ratio of crack width predicted to the experimental value is obtained as 0.990.

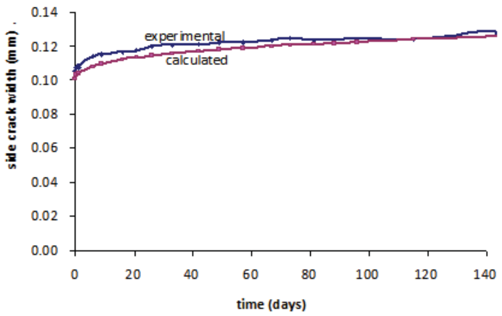



Fig. 9 Comparison of proposed model with test results of Lutz (doubly reinforced beam - Bottom crack width)

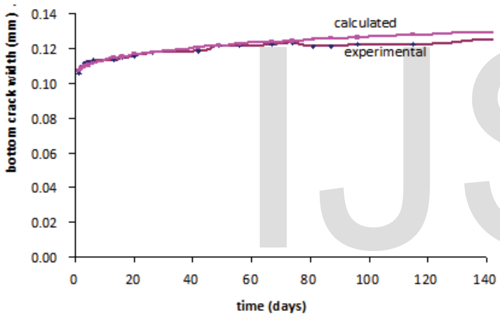



Fig. 10 Comparison of proposed model with test results of Lutz (doubly reinforced beam - Side crack width)

## V. SUMMARY AND CONCLUSIONS

An analytical method has been presented for the determination of width of cracks of R.C.C flexural members subjected to long-term sustained loading. Using the model cracking moment, strains and stresses of any layer at any time *t*, instantaneous and total curvature at any time *t*, spacing of cracks, crack width at any time *t*, etc can be determined. The model proposed was explained with a numerical example. The proposed analytical model has been compared satisfactorily with the test results available in the literature. The average ratio of $W_{cal}/W_{exp}$ was found to be 0.992. Creep, shrinkage and tension reinforcement ratio were found to be the main parameters affecting the magnitude of long-term crack width.

The analytical method presented in this paper is based on the forces applied on an R.C.C section calculated using the conditions of equilibrium. Therefore, this method can be extended to include partially prestressed beams as the axial force is also incorporated in the equilibrium equations.

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